M.Sc. Physics (Previous)

PAPER - II : CLASSICAL MECHANICS AND STATISTICAL MECHANICS

SYLLABUS

CLASSICAL MECHANICS

UNIT - I

Mechanics of a particle : Mechanics of a system of particles constraints. Principle of virtual work; D' Alembert's principle Lagrange's equations. Velocity dependent, potential Dissipation function. Hamilto's Principle, Derivation of Lagrange's equations from Hamilton's principle. Extension to non conservative and non - holonomic systems. Hamilton's equations of motion and their derivation from Lagrange's equation. Cyclic coordinates. Conservation theorems. Principal of least action.

Co-ordinates of a rigid body. Eulerian angles. Transformation matrix, Infinitesimal rotations. Representation as vectors Rate of change of a vector in a moving frame of reference. Centripetal acceleration, Coriolis force.

Angular momentum and kinetic energy of a rotating rigid body. The inertia tensor and moment of inertia. Euler equation of motion. Torque free motion of rigid body. Heavy symmetrical top with one point fixed.

UNIT II

Special theory of relativity. Lorentz transformation - Covariant four dimensional formulations : Force and energy equations in relativistic mechanics. Lagrangian formulation of relativistic mechanics. Lagrangian formulations. Canonical transformations. Generating functions. Simple examples of Lagrange and Poisson brackets. Their canonical invariance.

Hamilton - Jocobi equations. Hamilton's principle and characteristic functions. Separation of variables, Action - Angle variables, Kepler problem.

Formulation of the small oscillation problem. Eigen value equation and principle axis transformation. Frequencies of free vibration and normal co-ordinates. Examples of a linear triatomic molecule. Forced vibrations Effect of dessipative force.

TEXT BOOK :

1. Classical Mechanics by Herbert Goldstein (Narosa)
2. Classical Mechanics by Gupta, Kumar & Sharma (Pragathi Prakashan)

STATISTICAL MECHANICS

UNIT - III

Classical Statistical Mechanics : The Postulate of classical mechanics, Micro canonical ensemble, Derivation of thermodynamics, Equi-partition theorem.

Classical ideal gas (on the basis of micro canonical ensemble) Derivation of entropy for classical ideal gas and Gibb's paradox.

Canonical ensemble and grand canonical ensemble : Canonical ensemble, energy fluctuations in the canonical ensemble, Grand canonical ensemble, Density. fluctuations in the grand canonical ensemble, Equivalence between the canonical ensemble and grand canonical ensemble.

UNIT - IV

Quantum Statistical Mechanics : The Postulates of Quantum Statistical Mechanics, third law of thermodynamics. The ideal gases : Determination of thermodynamic parameters (micro canonical ensemble and grand canonical ensemble), foundations of statistical mechanics.

The partition function : Darwin - Fowler Method, classical limit of the partition function and the variational principle.

Ideal Fermi Gas : Equation of state of an ideal Fermi gas. Theory of white dwarf stars.

Ideal Bose gas : Photons, Phonons, Bose - Einstein condensation.

TEXT BOOK :

1. Statistical Mechanics by Kerson Huang, Wiley Eastern (pvt) ltd, New Delhi.

M.SC PHYSICS DISTANCE MODE SYLLABUS APPROVED IN THE P.G.B.O.S.PHYSICS MEETING HELD ON 22-12-2003

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Paper II : CLASSICAL MECHANICS AND STATISTICAL MECHANICS

MODEL QUESTION PAPER

Time : 3 Hrs | Max Marks : 100

Answer all questions. All questions carry equal marks.

1. a) What are different types of contraints encountered in dealing with motion of bodies? Derive Lagrange's equations of motion from D' Alemberts principle.

   Or

   b) Deduce Euler's equations of motion of a rigid body with one point fixed. What is Inertia tensor? Discuss its properties.

2. a) What are Poisson brackets? Mention their properties. Show that poisson brackets are invariant under a canonical transformation.

   Or

   b) State the basic postulates of the special theory of relativity and deduce Lorentz transformations. What are consequences of Lorentz transformations?

3. a) Distinguish between three types of ensembles. Explain Gibb's paradox. How do you resolve it?

   Or

   b) Explain a canonical ensemble. Show the equivalence between the canonical and micro canonical ensemble by studying the energy fluctions in the canonical ensemble.

4. a) Discuss about the postulates of Quantum statistics. Derive the expression for Sucker - Tetrode equation for the ideal Bose gas on the basis of micro canonical ensemble.

   Or

   b) Derive the Fermi - Dirac distribution function and show why an electron gas follows this statistics. Hence obtain the Fermi energy of an electron gas.

5. Write notes on any TWO of the following :

   a) Action - Angle variable and Kepler problem

   b) Frequencies of free vibrations in a linear tri atomic molecule

   c) Euler's angles

   d) Theory of White dwarf stars

   e) Third law of Thermodynamics.

CONTENTS

CLASSICAL MECHANICS

STATISTICAL MECHANICS

CLASSICAL MECHANICS

Classical Mechanics Part
Unit -I
Lesson 1

MECHANCS OF SYSTEM OF PARTICLES

Objective: In this lesson we learn about

1) Principles of mechanics of particles
2) Conservation laws
3) Constraints

Structure:

1.1 Mechanics of a particle
1.2 Conservation of energy
1.3 Mechanics of system of particles
1.4 Conservation of linear momentum for system of particles
1.5 To prove conservation of total Energy
1.6 Constraints
1.7 Types of constraints.

Introduction: Mechanics is the study of motion of physical bodies. Classical mechanics deals with the situations involving all bodies we come across in our daily life. However when the particles are too small or when they are travelling with velocities comparable to that of light, it was found that normal classical laws do not hold good and we have to apply other types of mechanics like quantum mechanics and relativistic mechanics. Even systems for which quantum mechanics holds, they are modeled according to classical mechanics and transformed to quantum mechanical form for further analysis. The progress of various sciences like astronautics, stellar dynamics, robotics, aerodynamics, and various branches of engineering still depend heavily on the foundation of classical mechanics. Hence it is very important for every student of physics to have considerable knowledge in classical mechanics and how the classical ideas were extended into the realm of quantum mechanics.

1.1 : Mechanics of a particle:

The position of any particle can be represented with three coordinates like x,y,z called Cartesian coordinates When the particle is in motion its position changes with time. The general displacement can be resolved intc components along x,y and z axes. As we already knew, the rate of change of displacement with time is callec velocity. If the particle possesses a mass m it is said to possess momentum. Newton formulated the laws o motion for a particle in motion. These laws are known as Newton's laws of motion are stated as

First law: A body continues in its state of rest or uniform motion, unless not disturbed by some external influence. It is also called Law of Inertia.

Second law: The time-rate of change of momentum is proportional to the impressed force. It is also called Law of Force.

Third law: To every action there is always equal and opposite reaction. This is known as Law of action and reaction.

: From Newton's second law of motion

$$\mathbf{F}=\frac{d}{dt}(m\,\mathbf{v})=\frac{d\mathbf{p}}{dt} \qquad ...(1)$$

When the total force $\mathbf{F}$ is zero, then $\frac{d\mathbf{p}}{dt}=0$, so the linear momentum is conserved ie. When a particle is moving in zero force field its linear momentum remains as constant. This is called law of Conservation of linear momentum

A particle motion need not be confined to a direction. The particles may be revolving around a centre of attraction (Central force field). In this case also the particle possess momentum, but, as the direction may change continuously, it is called angular momentum.

Consider a particle of mass 'm' and linear momentum $\mathbf{p}$ at position vector $\mathbf{r}$ relative to the origin 'O', of an inertial frame of reference. The moment of momentum is defined as the angular momentum. $\therefore\ \mathbf{L}=\mathbf{r}\times\mathbf{p}$

$\therefore$ Torque about the origin 'O' is $\mathbf{N}=\mathbf{r}\times\mathbf{F}$

$$\therefore \quad \mathbf{N}=\mathbf{r}\times\frac{d\mathbf{p}}{dt}=\frac{d}{dt}(\mathbf{r}\times\mathbf{p})-\frac{d\mathbf{r}}{dt}\times\mathbf{p}$$

$$=\frac{d}{dt}(\mathbf{r}\times\mathbf{p})-(\mathbf{v}\times m\,\mathbf{v})$$

$$\therefore \quad \mathbf{N}=\frac{d}{dt}(\mathbf{r}\times\mathbf{p})=\frac{d\mathbf{L}}{dt}$$

So the rate of change of angular momentum is equal to the Torque acting on the particle.

If the total torque $\mathbf{N}=0$, then $\frac{d\mathbf{L}}{dt}=0 \Rightarrow \mathbf{L}=$ constant. Hence the angular momentum is conserved in the absence of an external torque. This is called the law of Conservation of angular momentum

1.2 Conservation of energy: A force, which is derivable from scalar potential energy function in the manner $F=-\Delta V$ then it is called conservative. If the forces, which are conservative act upon the particle, then the total energy of the particle $(KE+PE)$ is conserved.

Under the action of such force let the particle moves from position 1 to position 2. Then workdone by the particle be,

$$W_{12}=\int_1^2 \mathbf{F}.d\mathbf{r}=\int_1^2 \frac{d\mathbf{p}}{dt}.d\mathbf{r}$$

But $\quad \mathbf{p}=m\dot{\mathbf{r}}$ and $d\mathbf{r}=\dot{\mathbf{r}}\,dt$

$$\therefore \quad W_{12}=\int_1^2 \frac{d}{dt}(m\dot{\mathbf{r}}).\dot{\mathbf{r}}\,dt$$

$$=\int_1^2 \frac{d}{dt}\left(\frac{1}{2}m\dot{\mathbf{r}}^2\right)dt=T_2-T_1 \qquad ...(3)$$

Where $T_1=KE$ of the particle at the position 1. $T_2=KE$ of the particle at the position 2.

We have $F=-\nabla v$

$$\therefore \quad W_{12}=\int_1^2 -\nabla V.d\mathbf{r}$$

$$=\int_1^2 -\frac{dV}{dr}dr=-\int_1^2 dV=V_1-V_2 \quad ...(4)$$

from (3) & (4) : $T_2 - T_1 = V_1 - V_2 \Rightarrow T_1 + V_1 = T_2 + V_2$

$\Rightarrow$ $T + V$ = constant. So the total energy of the particle is conserved even when it is moving in a conservative force field.

Till now we dealt with the motion of a single particle. But, in real situations we have to deal with not a single particle but a system of particles either independent or interacting with one another and moving under some system constraints. We now try to understand the relevant principles.

1.3.Mechanics of system of particles:

Consider a system of 'n' particles. The equation of motion interms of Newton's second law can be easily written as,

$$m_i \mathbf{a}_i = \dot{\mathbf{p}}_i = \mathbf{F}_i^j + \sum_{j \neq i} \mathbf{F}_i^j \qquad (\because i = 1,2,3....N) \qquad ...(1)$$

$\mathbf{F}_i^{(e)}$ = external force acting on i^{th} particle.

$\mathbf{F}_i^j$ = internal force on the i^{th} particle due to j^{th} particle.

All the particles of the system exert forces on one another. So the internal force on i^{th} particle must be the sum of forces due to all other particles is $\sum_{j=1}^{n} F_i^j$ excluding the term $j = i$, (Because from the definition $F_i^i = 0$)

Form Newton's third law the force $\mathbf{F}_i^j$ must be equal and opposite in direction to the force $\mathbf{F}_j^i$.

$$\therefore \qquad F_i^j = F_j^i \qquad ...(2)$$

Hence the internal forces occur in pairs and act along the line joining the two particles.

By considering all the particles of the system, the equation of motion of the total system is,

$$\sum_i \dot{p}_i = \frac{d^2}{dt^2} \sum_i m_i \bar{\mathbf{r}}_i$$

$$= \sum_i F_i^{(e)} + \sum_{i,j}^{1} F_i^j = \sum_1^1 F_i^{(e)} \quad ...(3)$$

we have $\sum_{i,j}^{1} F_i^j = \sum_{i,j}^{1} F_j^i = \frac{1}{2} \sum_{i,j}^{1} [\mathbf{F}_i^j + \mathbf{F}_j^i] = 0$

$$\left(\sum_{i,j}^{1} \text{means } i = j \text{ term is excluded} \right)$$

Let $\mathbf{R}$ be the position vector of center of mass.

$$\therefore \qquad \mathbf{R} = \frac{\sum_i m_i \bar{\mathbf{r}}_i}{\sum_i m_i} = \frac{\sum_i m_i \bar{\mathbf{r}}_i}{M}$$

Where $M = \sum_i m_i$ is the total mass of the system.

$$\therefore \qquad (3) \Rightarrow M \frac{d^2 \mathbf{R}}{dt^2} = \sum_i \mathbf{F}_i^{(e)} = \mathbf{F}^{(e)} \quad ...(4)$$

Total linear momentum of the system

$$\mathbf{P}=\sum_i m_i \dot{\mathbf{r}}_i$$

$$=\frac{d}{dt}\sum_i m_i \bar{\mathbf{r}}_i = M\dot{\mathbf{R}} \qquad ...(5)$$

1.4: Conservation of angular momentum for system of particles:

The total angular momentum of the system of particles $\mathbf{L}=\sum_i(\mathbf{r}_i\times\mathbf{p}_i)$

$$\therefore \quad \frac{d\mathbf{L}}{dt}=\frac{d}{dt}\sum_i(\mathbf{r}_i\times\mathbf{p}_i)=\sum_i[(\mathbf{r}_i\times\dot{\mathbf{p}}_i)+(\dot{\mathbf{r}}_i\times\mathbf{p}_i)]$$

$$=\sum_i(\mathbf{r}_i\times\dot{\mathbf{p}}_i) \qquad (\because \dot{\mathbf{r}}_i\times\mathbf{p}_i=\mathbf{O})$$

$$\Rightarrow \quad \dot{\mathbf{L}}=\sum_i \mathbf{r}_i\times\mathbf{F}_i^{(e)}+\sum_{i,j}^{1}\mathbf{r}_i\times\mathbf{F}_i^{j} \qquad ...(7)$$

The second term is due to the sum of internal torques. If the interacting forces are Newtonian in nature, the second term vanishes.

$$\therefore \quad \frac{d\mathbf{L}}{dt}=\mathbf{N}^{(e)}=\sum_i\mathbf{r}_i\times\mathbf{F}_i^{(e)} \qquad ...(8)$$

When there is no external torque $\mathbf{N}^{(e)}=0$.

$$\therefore \quad \frac{d\mathbf{L}}{dt}=0\Rightarrow \mathbf{L}=\text{Constant}. \qquad ...(9)$$

This is the law of conservation of angular momentum for a system of particles.

1.5. The law of conservation of total energy: For a system of particles $V=V^{(e)}+V^{\text{int}}$ ...(10)

and also, $\mathbf{F}_i^{(e)}=-\nabla_1 V^{(e)}$ and $\mathbf{F}_i^{\text{int}}=-\nabla_1 V^{\text{int}}$ ...(11)

Here V^{int} is the internal PE function.

$$V^{\text{int}}=\sum_j^{1}V_i^{j}=\sum_j^{1}V_j^{i}(|\mathbf{r}_i-\mathbf{r}_1|)=\sum_j^{1}V_i^{j}(|\mathbf{r}_j-\mathbf{r}_i|)$$

$$=\sum_i^{1}V_j^{i}(|\mathbf{r}_i-\mathbf{r}_j|)=V_j^{\text{int}}$$

Total PE $V^{\text{int}}=\sum_i V_i^{\text{int}}=\frac{1}{2}\sum_{i,j}^{1}V_j^{i}(|\mathbf{r}_i-\mathbf{r}_i|)$

The factor $\frac{1}{2}$ is taken because, while summing the mutual potential energies, a pair of particles i,j appear twice.

Proof:

Consider a system of particles in which i^{th} particle is displaced through $d\bar{\mathbf{r}}_i$ by the force $\mathbf{F}_i$. Then the amount of work done on the particle is,

$$m_i \ddot{\mathbf{r}}_i . d\mathbf{r}_i = \mathbf{F}_i . d\mathbf{r}_i = \left(\mathbf{F}_i^{(e)} + \sum_j \mathbf{F}_i^j\right) . d\mathbf{r}_i \qquad \ldots(12)$$

But $d\mathbf{r}_i = \dot{\mathbf{r}}_i . dt$.

$$\therefore \quad (12) \Rightarrow m_i \ddot{\mathbf{r}}_i \dot{\mathbf{r}}_i dt = \mathbf{F}_i^e . d\mathbf{r}_i + \sum_j \mathbf{F}_i^j . d\mathbf{r}_i$$

Hence for a system of particles,

$$\frac{d}{dt}\sum_i\left(\frac{1}{2}m_i\dot{\mathbf{r}}_i^2\right)dt = \sum_i \mathbf{F}_i^{(e)} . d\mathbf{r}_i + \sum_{i,j}^{1} \mathbf{F}_i^j . d\mathbf{r}_i \quad \ldots(13)$$

Consider $\sum_{i,j}^{1} \mathbf{F}_i^j . d\mathbf{r}_i = \frac{1}{2}\sum_{i,j}^{1}\left(\mathbf{F}_i^j . d\mathbf{r}_i + \mathbf{F}_j^i . d\mathbf{r}_j\right) \ldots (14)$

For the forces obeying Newton's third law, $\mathbf{F}_i^j = -\mathbf{F}_j^i$

And also, $V_i^j = V_i^j\left(|\mathbf{r}_i - \mathbf{r}_j|\right)$ and $V_i^j = V_j^i$

$\therefore \ \mathbf{F}_i^j = -\nabla_i V_i^j, \mathbf{F}_j^i = -\nabla_j V_i^j$ and $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$

$$\therefore \ F_i^j = -\nabla_i V_i^j = -\frac{\partial}{\partial \mathbf{r}_i}\left(V_i^j\right) = -\frac{\partial}{\partial \mathbf{r}_{i,j}} V_i^j\left(\frac{\partial \mathbf{r}_{ij}}{\partial \mathbf{r}_i}\right)$$

$$= -\nabla_{ij} V_i^j . \left(\frac{\partial}{\partial \mathbf{r}_i}\left(\mathbf{r}_i - \mathbf{r}_j\right)\right) = -\nabla_{ij} V_i^j \quad \ldots(15)$$

we have $\mathbf{F}_j^i = -\nabla_j V_i^j = -\frac{\partial}{\partial \mathbf{r}_j}\left(V_i^j\right) = -\frac{\partial V_i^j}{\partial \mathbf{r}_{ij}}\left(\frac{\partial \mathbf{r}_{i,j}}{\partial \mathbf{r}_j}\right)$

$$= -\nabla_{i,j} V_i^j\left[\frac{\partial}{\partial \mathbf{r}_j}\left(\mathbf{r}_i - \mathbf{r}_j\right)\right] = \nabla_{i,j} V_i^j$$

$$\ldots(16)$$

Substituting (15), (16) in (14)

$$\sum_{i,j}^{1} \mathbf{F}_i^j . d\mathbf{r}_i = \frac{1}{2}\sum_{i,j}^{1}\left(-\nabla_{i,j} V_i^j . d\mathbf{r}_i + \nabla_{i,j} V_i^j . d\mathbf{r}_j\right)$$

$$= -\frac{1}{2}\sum_{i,j}^{1} \nabla_{i,j} V_i^j . d\left(\mathbf{r}_i - \mathbf{r}_j\right)$$

$$= -\frac{1}{2}\sum_{i,j}^{1} \nabla_{ij} V_i^j . d\mathbf{r}_{i,j} \qquad \ldots(17)$$

If the external force is also conserved,

$$F_i^{(e)} = -\nabla_i V_i^{(e)} \qquad \ldots(18)$$

from the equation (17) & (18) we can write (13) as,

$$\frac{d}{dt}\sum_i\left(\frac{1}{2}m_i\dot{\mathbf{r}}_i^2\right)dt=-\sum_i\nabla_iV_i^e.d\mathbf{r}_i-\frac{1}{2}\sum_{ij}^{1}\nabla_{ij}V_i^j.d\mathbf{r}_{ij}$$

On integration $\sum_i\frac{1}{2}m_i\dot{\mathbf{r}}_i^2=-\sum_i^{1}V_i^e-\frac{1}{2}\sum{}^{'}V_i^j+\text{constant}$

$$\Rightarrow\quad\sum_i\frac{1}{2}m_i\dot{\mathbf{r}}_i^2+\sum_iV_i^e+\frac{1}{2}\sum_{i,j}^{1}V_i^j=\text{constant}$$

$$\Rightarrow\quad T+V=\text{constant}\qquad\ldots(19)$$

where $V=\sum_i^{1}V_i^{(e)}+\frac{1}{2}\sum_{i,j}^{1}V_i^j$

V is the sum of external and internal potential energies of the system. Eq. (19) is the conservation of energy.

1.6 Constraints :

A motion, which cannot proceed arbitrarily in any manner, is called constrained motion. Motion along a specified path is an example of constrained motion. Depending on the nature of the motion of particle, the coordinates are considered to describe the motion. If the particle is free to move in space, then three coordinates are needed to describe its motion. Thus imposing constraints on a mechanical system is to simplify the configuration of the system.

Example 1: Consider a particle moving in space. It requires three coordinates to determine its position at any instant. If its movement is restricted on the surface of a sphere, there exists a relation between its coordinates as,

$$x^2+y^2+z^2=a^2\Rightarrow r=a.\qquad\ldots(1)$$

Eqn. (1) is the eqn. of a sphere with its center at origin. By using the equations of constraints a coordinate can be eliminated from the set of three coordinates. Instead of Cartesian coordinates if we express the problem interms of spherical polar coordinates (r, θ and ϕ), then the two coordinates θ and ϕ will be sufficient to describe the position of the particle completely.

Example 2: Consider the simple pendulum whose motion is confined in the vertical plane. To locate the position of the bob in motion only two coordinates are required. Let the motion of the bob taken placed under a constraint that the distance 'l' of the bob is to remain same at all time. This condition imposed by the constraint can be expressed in the form of an equation either between x and y or r and θ.

∴ Constraints $x^2+y^2=l^2$, or $r=l$ ... (2)

The eqn (2) is used to reduce the number of coordinates, which other wise would have been two.

Example 3. Consider a rigid body. It is defined as a system of particles in which the relative distance of the constituent particles are fixed and cannot vary with time. In this case constraints are expressed by equations of the form $r_{ij}=C_{ij}$. In this $r_{i,j}$ denote the distance between i^{th} and j^{th} particles and C_{ij} are constants. If $\mathbf{r}_i(x_i,y_i,z_i)$ and $\mathbf{r}_j(x_j,y_j,z_j)$ are the coordinates w.r.t. the origin, then the conditions can be expressed as,

$$(x_i-x_j)^2+(y_i-y_j)^2+(z_i+z_j)^2=(C_{ij})^2\quad\ldots(3)$$

In general, for a system of N particles,

$$f(x_1y_1z_1:x_2y_2z_2:x_ny_nz_n:t)=0\qquad\ldots(4)$$

1.7: Types of constraints:

1) Holonomic constraints 2) Non-Holonomic constraints

1. Holonomic Constraints: Constraints that can be expressed in the form of $f(X_1, Y_1, Z_1, X_2, Y_2, Z_2 \ldots\ldots\ldots X_n Y_n Z_n \ . t) = 0$
where time 't' may occur in case of constraints which may vary with time.

Ex : Let us consider the motion of a simple pendulum confined to move in the vertical plane. We would need only two co-ordinates (contain (X, Y, Z) or polar co-ordinates (r, θ) with respect to the point of suspension with 'O' as)

To locate the position of the bob in motion, however motion of the bob is not free but takes place under a constraint that the distance 'l' or the bob B to remain the same from 'O' at all times. This condition imposed by the constraint can be expressed in the form of an equation either by X and Y (or) r and θ

$$x^2 + y^2 = l^2$$

$$r = l \qquad \ldots(1)$$

In plane polar co-ordinates the equation looks simpler again one co-ordinate either X or Y in Cartesian co-ordinates, r (or) θ in polar co-ordinates would be sufficient to describe the motion. Now that we have utilized eq. (1) to reduce the number of co-ordinates which otherwise would have been two.

2. Non-Holonomic Constraints :

Constraints, which are not expressible in the form, $f(X_1, Y_1, Z_1, X_2, Y_2, Z_2 \ldots\ldots X_n Y_n Z_n \ . t) = 0$ are termed as non-holonomic constraints.

The motion of the particle placed on the surface of sphere under the action of gravitational force is bound by non-holonomic constraints, for it can be expressed as an inequality.

$$r^2 - a^2 \geq 0$$

Equality sign holds until the particle rolls on the sphere and when it leaves the sphere we must have

$$r^2 - a^2 > 0$$

Ex : The constraints involved in the motion of the molecules in a gas container are non-holonomic.

Scleronomic & Rheonomic Constraints:

If the constraints are independent of time they are termed as scleronomic but if they contain time explicitly they are called Rheonomic.

A bead sliding on a moving wire is an example of Rheonomic constraint.

Force of constraints:

Constraints not only interfere with the solution or the problems in that the co-ordinates are no longer independent but they are always associated with the forces by virtue of which they restrict the motion of the system. Such forces are termed as forces of constraints. We generally formulate the laws of mechanics in a way that the work done by the forces of constraints is zero, when the system is in motion.

Forces of constrains in the case of a bead sliding on a wire is the reaction by the wire exerted on the bead at each point. The surface of sphere exerts a reaction force on the particle normally at each point.

Summary: For a particle moving in a force free space the linear momentum and energy are conserved. When a particle is moving in conservative force field linear momentum, angular

momentum, and total energy of the system are conserved. These conservation laws hold good even for a system of particles moving in a conservative force field. In general in a multi-particle system a particle may have specific relation ship with its neighbors and so cannot move arbitrarily. We say that its motion is constrained. The relations, which restrict the motion of particles, are called constraints. The constraints are divided in to holonomous and non-holonomous types. Non-holonomous constraints are further divided into scleronomous and rheonomous constraints. Constraints are associated with forces called constraint forces. However the laws of mechanics are so formulated so that the work done by the forces of constraints are zero.

Key Words:

Angular momentum, linear momentum, total energy (T+V), and constraints, Holonomic, Non-holonomic constraints, forces of constraints.

Self Assessment Questions:

1. What are constraints? Give specific examples to explain the forces of constraints.
2. Prove the laws of conservation of linear momentum, angular momentum and energy for a system of particles.
3. State and prove work-energy theorem.

Reference books:

1. Classical Mechanics: H.Goldstein
2. Mechanics: Simon
3. Mechanics: Gupta, Kumar and Sharma.

Classical Mechanics Part
Unit I
Lesson 2
Lagrangian Formulation

Objective: To learn about
1) The D'Alembert's Principle and derivation of Lagrangian equation of motion from it
2) The concept of virtual displacement and virtual work.
3) Hamilton's variational principle and deriving Hamilton's canonical equations of motion.
4) Conservation theorems.
5) Principle of Least Action.

Structure:
2.1 D'Alembert's Principle
2.2 Lagrange's equations
2.3 Virtual displacement and work
2.4 Principle of virtual work
2.5 Hamilton's approach
2.6 Hamiltons variational principle
2.7 Canonical equations of motion
2.8 Conservation theorems
2.9 Principle of least action.

2.1: D'Alembert's Principle

This method is based on the principle of virtual work. The system is subjected to an infinitesimal displacement consistent with the forces and constraints imposed on the system at the given instant 't'. This change in the configuration of the system is not associated with a change in time (i.e.) there is no actual displacement during which forces and constraints may change and hence the displacement is termed as virtual displacement.

Now suppose the system is in equilibrium (i.e.) the total force F_i on every particle is zero. Then work done by this force in a small virtual displacement δr_i will also vanish i.e.,

$$\sum_i F_i \cdot \delta r_i = 0$$

Let this total force be expressed as sum of applied force F_i^a and forces of constraints f_i then above equation takes the form

$$\sum_i F_i^a . \delta r_i + \sum_i F_i . \delta r_i = 0$$

We now consider the systems for which the virtual work of the forces of constraints is zero. An example of such a system can be had in mind is that a particle i.e. constrained to move on a smooth surface. So that the forces of constraints being $\perp$ to the surface while virtual displacement is tangential to it. Then virtual work done by forces of constraints is zero. Thus

$$\sum_i F_i^a \delta r_i = 0 \qquad ...(1)$$

This equation. is termed as principle of virtual work. To interpret the equilibrium of the system D'Alembert adopted an idea of a reversed force. D'Alembert conceived that a system would remain in equilibrium under the action of a force equal to the actual force F_i.

F_i plus reversed affective force

$\dot{P}_i$ thus $F_i + (-\dot{P}_i) = 0$ (or) $F_i - \dot{P}_i = 0$

Thus the principle of virtual work takes the form

$$\sum_i (F_i - \dot{P}_i)\,\delta r_i = 0$$

Again writing $F_i = F_i^a + f_i$

$$\sum_i (F_i^a - \dot{P}_i)\,\delta r_i + \sum_i f_i . \delta r_i = 0$$

Dealing with the system for which the virtual work of the forces of constraints is zero. We write

$$\sum_i (F_i^a - \dot{P}_i)\,\delta r_i = 0$$

(Force of constraints is no more in picture)

It is better to drop subscript 'a'. Thus

$$\sum_i (F_i - \dot{P}_i).\delta r_i = 0 \qquad ...(2)$$

Which is called D'Alembert's principle. To satisfy eq. (2) we can't equate the coefficients of δr_i to zero. Since δr_i are not independent of each other and hence it is necessary to transform δr_i changes into the changes of G. Co-ordinates δq_j which are independent of each other. The coefficient of every δq_i will then equated to zero.

2.2: Lagrange's Equation from D'Alembert's Principle:

The co-ordinate transformation eq's are

$$r_i = r_i(q_1, q_2, \ldots\ldots q_{n_2}\ t) \qquad ...(1)$$

So that $\dfrac{dri}{dt} = \dfrac{\partial ri}{\partial q_1}\dfrac{dq_1}{dt} + \dfrac{\partial r_i}{\partial q_2}\dfrac{dq_2}{dt} + \ldots\ldots \dfrac{\partial r_i}{\partial t}\dfrac{dt}{dt}$

$$V_i = \sum_j \frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t} \qquad ...(3)$$

Further infinitesimal displacement δr_i can be connected with δq_j as

$$\delta r_i = \sum_i \frac{\partial r_i}{\partial q_j}\delta q_j + \frac{\partial r_i}{\partial t}\delta t$$

But last term is zero since in virtual displacement only co-ordinate displacement is considered and not that of time.

$$\therefore\ \delta r_i = \sum_j \frac{\partial r_i}{\partial q_j}\delta q_j$$

Now we write eq (2) as

$$\sum_i (F_i - \dot{P}_i)\,\delta r_i = 0$$

$$\sum_{ij} F_i \frac{\partial r_i}{\partial q_j} \delta q_j - \sum_{ij} \dot{P}_i \frac{\partial r_i}{\partial q_j} \delta q_j = 0$$

we define $\sum_i F_i \frac{\partial r_i}{\partial q_j} = Q_j$

(The components of generalized force)

As discussed under generalized force q's need not have dimensions of length and similarly it is not necessary for the Q's to have the dimensions of force. But it is necessary that the product $Q_j \ \delta q_j$ must have the dimensions of work. Thus above eq. takes the form

$$\sum_j Q_j \delta q_j - \sum_{ij} \dot{P}_i \left(\frac{\partial r_i}{\partial q_j} \right) \delta q_j = 0 \quad \ldots(4)$$

Let us evaluate the second term of eq. (4)

$$\sum_{ij} \dot{P}_i \left(\frac{\partial r_i}{\partial q_j} \right) \delta q_j = \sum_{ij} m_i \ddot{r}_i \left(\frac{\partial r_i}{\partial q_j} \right) \delta q_j$$

$$= \sum_{ij} \left\{ \frac{d}{dt} \left(m_i r_i \frac{\partial r_i}{\partial q_j} \right) - m_i r_i \frac{d}{dt} \left(\frac{\partial r_i}{\partial q_j} \right) \right\} \delta q_j$$

$$= \sum_{ij} \left\{ \frac{d}{dt} \left(m_i V_i \frac{\partial r_i}{\partial q_j} \right) - m_i V_i \frac{d}{dt} \left(\frac{\partial r_i}{\partial q_j} \right) \right\} \delta q_j \quad \ldots(5)$$

Further

$$\frac{d}{dt} \left(\frac{\partial r_i}{\partial q_j} \right) = \sum_k \frac{\partial}{\partial q_k} \left(\frac{\partial r_i}{\partial q_j} \right) \frac{\partial q_k}{\partial t} + \frac{\partial}{\partial t} \left(\frac{\partial r_i}{\partial q_j} \right) \frac{dt}{dt}$$

$$= \sum_k \frac{\partial^2 r_i}{\partial q_k \partial q_j} \dot{q}_k + \frac{\partial^2 r_i}{\partial t . \partial q_j}$$

$$= \sum_k \frac{\partial}{\partial q_j} \left(\frac{\partial r_i}{\partial q_k} \dot{q}_k + \frac{\partial r_i}{\partial t} \right)$$

$$= \frac{\partial}{\partial q_j} \left(\frac{dr_i}{dt} \right) = \frac{\partial V_i}{\partial q_j} \quad \ldots(6)$$

Also differentiating eq. (3) write $\dot{q}_j$ we get

$$\frac{\partial V_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j} \quad \ldots(7)$$

Putting eq. (6) and (7) in eq. (5), we get

$$\sum_{ij} \dot{P}_i \left(\frac{\partial r_i}{\partial q_j} \right) \delta q_j = \sum_{ij} \left\{ \frac{d}{dt} m_i V_i \left(\frac{\partial V_i}{\partial \dot{q}_j} \right) - m_i V_i \left(\frac{\partial V_i}{\partial q_j} \right) \right\} \delta q_j = \sum_j \left\{ \frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_j} \left(\sum_i \frac{1}{2} m_i V_i^2 \right) \right] - \frac{\partial}{\partial q_j} \left(\sum_i \frac{1}{2} m_i V_i^2 \right) \right\} \delta q_j$$

With this substitution eq. (4) becomes

$$\sum_j Q_j \delta q_j - \sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\delta q_j = 0$$

Where for $\sum_i \frac{1}{2} m_i V_i^2$ 'T' is written since it represents the total $K.E.$ of the system further

$$\sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j\right]\delta q_j = 0$$

Since the constraints are holomonic.
q_j are independent of each other and hence to satisfy above eq. the coefficient of each δq_j should separately vanish (i.e.)

$$\left[\sum_j \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] = Q_j \quad ...(8)$$

As 'j' from 1 to n there will be 'N' such second order eq's.

2.3:Virtual Displacement:

Any imaginary displacement, which is considered with the constrained relation at any given instant (The real time doesn't change) is, called virtual displacement.

Virtual work:
Work done by the forces due to virtual displacement is called virtual work.
Virtual work $\delta W = \sum_i F_i \delta r_i$

Here $F_i = F_1^{(a)} + f_i$

Where F_i^a is applied force and f_i is forces of constraints
'δr_i' is displacement of position co-ordinate only
It does not involve variation of time

$$\delta r_i = \delta r_i (q_1, q_2 ......q_n)$$

2.4 Principle of virtual work :

Suppose the system is in equilibrium the resultant force on any particle vanishes
(i.e.) $F_i = 0$ for all values of 'i'.
Now virtual work

$$\delta w_i = \sum_i F_i \delta r_i = 0$$

$$\sum_i F_i \delta r_i = 0$$

$$\sum_i F_i^{(a)} \delta r_i + \sum_i f_i \delta r_i = 0$$

If the virtual work done by the force of constraints is zero;
(i.e.) $\sum_i f_i \delta r_i = 0$ then virtual work done by the applied force on a system in equilibrium state vanishes.

(i.e.) $\delta w_i = \sum_i F_i^{(a)} \delta r_i = 0$

The necessary condition for state equilibrium is that the virtual work done by all the constrained forces vanishes. This is called the principle of virtual work.

2.5 Hamiltonian Approach

In the Lagrangian formulation, the equations of motion for a system are obtained in terms of generalized coordinates and generalized velocities. These equations of motion form a set of second order differential equations. Hamilton proposed an alternative formulation by using generalized coordinates and generalized momenta. This formulation results in two sets of first order differential equations. Both the formulations are identical, but the Hamiltonian formulation is more fundamental to the foundations of statistical and quantum mechanics. This formulation is particularly useful when some of the generalized momenta are the constants of motion.

Hamiltonian observed ambiguity of Lagrangian function for which it consists of derivative co-ordinates $(\dot{q}_j)$and underivative co-ordinates are not given equal status (or) equal flooding. Some co-ordinates are not derivative co-ordinates and some other co-ordinates are underivative co-ordinates.
Hamiltonian simply place momenta co-ordinate in the case of $\dot{q}_j$ co-ordinate and the entire co-ordinates are given equal status. The new function would be written as $H(q_j, P_j, t)$
$H(q_j, p_j, t)$ Hamiltonian quantity is nothing but energy quantity and it is sum of $K.E$ and $P.E$

$\therefore \quad H = T + V$

According to Hamiltonian approach if any particle moves in free space requires '6' co-ordinates (3 are Generalized co-ordinates and rest of '3' are momenta co-ordinates). If 'N' particle body moves in the free space it requires 6N co-ordinates. Out of '6N' co-ordinates 3N are G. co-ordinates and remaining 3N are momenta co-ordinates. The 'N' particle body reduced to single particle known as *system point*. This system point can travel through '6N' dimensional space. Then the space is known as *phase space*.

2.6 Hamilton variational principle

The principle states that the integral $\int_1^2 (T - V)dt$ shall have a stationary value (extremum). Where T is the $K.E$ of the mechanical system. It is a function of position co-ordinates and their derivatives. V is the P.E. of the mechanical system, it is a function of co-ordinates only. Such a system for which V is purely a function of co-ordinates is called conservative system.

Statement : Hamilton's principle for conservative system is stated as follows.
The motion of the system from time 't_1' to time t_2 is such that the line integral

$I = \int_{t_1}^{t_2} L dt$ where

$L = T - V$ is an extremum for the path of motion.

Deduction: Let us consider a conservative system of particles employing the generalized co-ordinates. The integral can be written as

$$\int_{t_1}^{t_2} [T(q_j, \dot{q}_j) - V(q_j)] dt$$

∴ According to Hamilton's principle we have

$$\delta\int_{t_1}^{t_2}\left[T(q_j,\dot{q}_j)-V(q_j)\right]dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2}\sum_j\left[\left(\frac{\partial T}{\partial q_j}\delta q_j+\frac{\partial T}{\partial \dot{q}_j}\delta\dot{q}_j\right)-\frac{\partial V}{\partial q_j}\delta q_j\right]dt=0$$

$$\Rightarrow \int_{t_1}^{t_2}\sum_j\left(\frac{\partial T}{\partial q_j}-\frac{\partial V}{\partial q_j}\right)\delta q_j\,dt+\int_{t_1}^{t_2}\sum_j\frac{\partial T}{\partial \dot{q}_j}\delta\dot{q}_j\,dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2}\sum_j\left(\frac{\partial T}{\partial q_j}-\frac{\partial V}{\partial q_j}\right)\delta q_j\,dt+\int_{t_1}^{t_2}\sum_j\frac{\partial T}{\partial \dot{q}_j}\frac{d}{dt}(\delta q_j)dt = 0$$

Integrating by parts the second term we get

$$\Rightarrow \int_{t_1}^{t_2}\sum_j\left(\frac{\partial T}{\partial q_j}-\frac{\partial V}{\partial q_j}\right)\delta q_j\,dt+\left[\sum_j\frac{\partial T}{\partial q_j}\delta q_j\right]_{t_1}^{t_2}-\int_{t_1}^{t_2}\sum_j\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\delta q_j dt = 0$$

Since in such a variation there is no co-ordinate variation at end points,

$$\delta q_j\Big|_{t_1}^{t_2} = 0$$

Hence eq. Reduces to

$$\int_{t_1}^{t_2}\sum_j\left(\frac{\partial T}{\partial q_j}-\frac{\partial V}{\partial q_j}\right)\delta q_j\,dt-\int_{t_1}^{t_2}\sum_j\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\delta q_j\,dt = 0$$

$$\int_{t_1}^{t_2}\sum_j\left[\frac{\partial T}{\partial q_j}-\frac{\partial V}{\partial q_j}-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\right]\delta q_j\,dt = 0$$

Since each δq_j are independent of each other, the coefficient of every δq_j should be equated to zero to satisfy above equation.

$$\therefore\left[\frac{\partial T}{\partial q_j}-\frac{\partial V}{\partial q_j}-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\right]=0$$

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial}{\partial q_j}(T-V)\right]=0$$

Part now for conservative systems 'V' is not a function of velocity '$\dot{q}_j$' but only of co-ordinates

$$\therefore\left[\frac{d}{dt}\left(\frac{\partial(T-V)}{\partial \dot{q}_j}\right)-\frac{\partial(T-V)}{\partial q_j}\right]=0$$

Here we introduce the concept of scalar function called Lagrangian 'L' for a conservative system and is equal to $(T-V)$ thus the above equation then takes the form

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}\right]=0$$

Thus set of eq's. is called Lagrangian eqs.of motion.

2.7 Hamilton's canonical equations of motion

In Hamiltonian postulation we provide generalized momenta, an independent status placing it on equal footing with the G.co-ordinates. Hamiltonian is then to be regarded in general as a function of the position co-ordinates q_j, the momenta P_j and the time 't' (i.e.)

$$H = (q_j, P_j, t)$$

The differential of 'H' gives

$$dH = \sum_j \frac{\partial H}{\partial q_j} dq_j + \sum_j \frac{\partial H}{\partial P_j} dP_j + \frac{\partial H}{\partial t} dt \quad ...(1)$$

Further 'H' is defined as

$$H = \sum_j P_j \dot{q}_j - L$$

So that

$$dH = \sum_j \dot{q}_j dP_j + \sum P_j d\dot{q}_j - dL \quad ...(2)$$

But Lagrangian is

$$L = L(q_j, \dot{q}_j, t)$$

So that

$$dL = \sum_j \frac{\partial L}{\partial q_j} dq_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j + \frac{\partial L}{\partial t} dt \quad ...(3)$$

Putting the value of dL from eq. (3) in eq. (2), we get

$$dH = \sum_j \dot{q}_j dP_j + \sum_j P_j d\dot{q}_j - \sum_j \frac{\partial L}{\partial q_j} dq_j - \sum_j \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j - \frac{\partial L}{\partial t} dt \quad ...(4)$$

Recognising $\frac{\partial L}{\partial \dot{q}_j} = P_j$ and $\frac{\partial L}{\partial q_j} = \dot{P}_j$ and putting these in eq. (4), we get

$$dH = \sum_j \dot{q}_j dP_j + \sum_j P_j d\dot{q}_j - \sum_j \dot{P}_j dq_j - \sum_j P_j d\dot{q}_j - \frac{\partial L}{\partial t} dt$$

$$dH = \sum_j \dot{q}_j dP_j - \sum_j \dot{P}_j dq_j - \frac{\partial L}{\partial t} dt \quad ...(5)$$

Comparing coefficients in eq. (5) and eq. (1), we get

$$\left.\begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial P_j} \\ \dot{P}_j &= -\frac{\partial H}{\partial q_j} \end{aligned}\right\} \quad ...(6)$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \quad ...(7)$$

eq. (6) is known as Hamilton's canonical eqs. of motion.